Static and Dynamic Analysis of Mooring Lines

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A general, computationally efficient approach for treating mooring line static and dynamic response problems is presented. The technique is a form of incremental linearization. An initial equilibrium configuration is obtained from explicit integration of the nonlinear equations including only the dominant loading. Other loadings are then considered as increments in the equations which are linearized for each load. The line solution or transfer matrix formulation is used to compute the small deformation due to each increment. Changes in mooring line properties and loads are accommodated.

Nomenclature

A = cross-sectional area of the mooring line
C_n = drag coefficient of the mooring line for flows normal to the line elements
$D(R,\Phi)$ = normal hydrodynamic load per unit length of the mooring line
E = modulus of elasticity of the mooring line
$F(R,\Phi)$ = tangential hydrodynamic load per unit length of the mooring line
h_i = unstretched length of the ith segment of the mooring line
m_{xx},m_{yy} = added mass components per unit length of mooring line
R = normal component of the hydrodynamic force arising from the current per unit length of the mooring line
S = unstretched length of the mooring line
t = time
T = tension of the mooring line
v = increment of the relative velocity
V = relative velocity of the water current
w = increment of buoyant weight of the mooring line
W = buoyant weight of the mooring line
(x,y) = increment of the components of the location
(X,Y) = coordinate components of the location
Δt = increment of the tension
ϕ = increment of the angle
Φ = angle between the tangent of the trajectory of the mooring line and the x axis

Introduction

ALMOST all offshore operations and deep ocean installations involve mooring lines. The reliability of such offshore moorings is an essential ingredient in the reliability of the installations and in the success of the operations. It is therefore very important that the performance of the mooring lines subject to the environmental forces in the ocean can be accurately predicted.

The literature contains many solutions for two-point and initial value cable problems. However, some important mooring line problems involve loading-dependent boundary conditions and unknown initial conditions. Many such problems are essentially one-point problems with unknown initial conditions for which cable solution methods encounter difficulties.

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Mooring line performance has been the subject of many recent investigations. Reviews of portions of the literature have been provided by Beateaux[1] and Cararella and Parsons.[2] Nearly all existing methods can be categorized into four general approaches: the explicit integration solutions, the incremental integration solutions, the finite element methods, and finite difference solutions. These solutions are limited by restrictions on the type of mooring lines that can be treated and/or on the type of external forces of the environment accepted. The primary sources of the difficulties can be attributed to the nonlinearity of the equations of motion and the fact that the boundary conditions are functions of the external loads and time.

The method presented here is an effective combination of the explicit integration solution, the line solution technique, and the incremental deformation approach for nonlinear structural analyses. Such a combination removes the difficulties encountered by the existing methods. The proposed method is appropriate for both two- and three-dimensional problems. For reasons of simplicity, the detailed discussions and formulations are limited to the two-dimensional problem.

The Problem and Existing Solution Methods

The equations of motion of the mooring line can be expressed in many forms depending on personal preference and the simplifying assumptions involved. It should be noted that many formulations for suspended cables and transmission lines are not suitable for mooring lines because of the assumption of high tension and large radius of curvature. The following formulation is an extension of Pode's equations.[3] This type of formulation, which is simpler than those using higher-order derivatives of the coor-

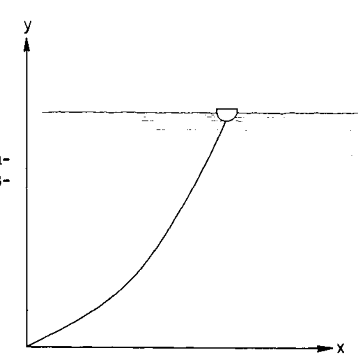

Fig. 1 A two-dimensional mooring system.

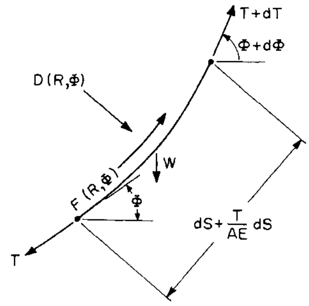

Fig. 2 A segment of cable.

dinate variables x and y, is commonly used in the general approaches to this problem.

Equations

A segment of the two-dimensional cable system of Fig. 1 is shown in Fig. 2 with common environmental forces. Equilibrium conditions applied to this stretched segment lead to

$$\frac{\partial T}{\partial S} = \left[W - \left(\frac{W}{g} + m_{yy}\right)\ddot{Y}\right]\sin\Phi - F(R,\Phi)\left(1 + \frac{T}{AE}\right) - \left(\frac{W}{g} + m_{xx}\right)\ddot{X}\cos\Phi$$

$$\frac{\partial \Phi}{\partial S} = \frac{1}{T}\left\{D(R,\Phi)\left(1 + \frac{T}{AE}\right) + \left[W - \left(\frac{W}{g} + m_{yy}\right)\ddot{Y}\right]\cos\Phi + \left(\frac{W}{g} + m_{xx}\right)\ddot{X}\sin\Phi\right\} \tag{1}$$

where the inertia forces, $m_{xx}\ddot{X} = m_{xx}\partial^2X/\partial t^2$, $m_{yy}\ddot{Y} = m_{yy}\partial^2Y/\partial t^2$, caused by the motion of the mooring line, are included. Here m_{xx}, m_{yy} are components of the added mass/length in the x and y directions. Other added mass components of second order have been neglected. It follows from geometric considerations of the segment of Fig. 2 that

$$\frac{\partial X}{\partial S} = \left(1 + \frac{T}{AE}\right)\cos\Phi \qquad \frac{\partial Y}{\partial S} = \left(1 + \frac{T}{AE}\right)\sin\Phi \tag{2}$$

Explicit Integration

Since Eqs. (1) and (2) are nonlinear, even for a static problem, a general explicit solution cannot be derived. References 1 and 3 contain comprehensive summaries of existing explicit static solutions to simplified forms of Eqs. (1). Dynamic problems are complicated and even more simplifying assumptions are required to permit explicit solutions. One commonly used assumption is that the tension throughout the cable is great and the deformation is small so that the nonlinear terms in the differential equation can be neglected. This leads to string theory. Numerous contributions, e.g., those in Refs. 4–6, have been made along this line for the problems of suspended structures, guy cables, and transmission lines. But string theory is not truly suitable for mooring line problems. However, Alexandrov[7] has applied this theory to the dynamics of mooring line problems. The result is valid for very restricted cases only.

Finite Element Method

The finite element method has been used successfully for many structural problems. It seems quite natural that this powerful method can also be applied with similar success to mooring line problems. This is not entirely true. First of all, the basic assumption of the usual finite element method is that the structure behaves linearly so that the principle of superposition is valid. Since mooring line problems are nonlinear, the use of ordinary finite element methods is not valid. Secondly, the finite element method is limited to problems with prescribed external forces and boundary conditions. For most mooring line problems, the external loads depend on the deformation whereas the boundary conditions depend on the external forces and time. For example, in the case of a single buoy mooring line of given length, both the position and force at the buoy end depend on the external loads.

Iterative procedures have been developed for related problems[8,9] in order to overcome the limitations of the ordinary finite element method. Skop and O'Hara[10] introduced a technique that employs static equilibrium and an iteration process for static problems.

Finite Difference Method

The finite difference method is more flexible than the finite element method for mooring line problems. It encounters the same difficulties in the position-dependent external forces and the time and external force-dependent boundary conditions, but similar iterative schemes can also be used. Walton and Polachek[11] developed a numerical method for the transient motions of submerged cables using finite differences and the Newton-Raphson iterative method. Their method is limited to problems with prescribed boundary conditions and initial positions. It is also limited to short cables because the elasticity of the cable is neglected. Nath and Felix[12] have developed a finite difference method for cables in deep water. Their method is also restricted to load-independent boundary conditions.

Incremental Integration

Incremental integration entails an approximate integration of the differential equations of motion using small elements of cable length. This approach is sometimes referred to as the numerical integration method, although "incremental integration" is more indicative of the procedure involved. Either given or estimated initial conditions are required. In the latter case, an iteration process is used to seek the correct solution. This kind of iteration can be very time-consuming and may even fail to converge for some cases, depending on the accuracy of the estimated initial values.

Using incremental integration, Reid[13] contributed probably the most general dynamic analysis of mooring lines. His equations are three-dimensional and include damping. But his method entails all of the limitations of incremental integration and encounters more than the usual difficulties because more variables and more computation steps are involved in each increment. No control over numerical errors or convergence checks are included in his formulation.

The incremental integration method is usually sufficiently accurate for mooring lines under high tension. When the line is long and the tension in some sections is small, the errors involved become significant.[13] Sometimes accuracy of the incremental integration approach can be improved by using an initial estimate obtained by explicit integration of a similar yet simpler configuration and by taking the concept of a critical angle[3] into account. In fact, if the initial values lie on the wrong side of the critical angle, incremental integration can never reach the correct solution.

The Present Approach

The method put forth here combines an incremental linearization with a line solution approach. The nonlinear equations are solved by an incremental linearization similar to those used for nonlinear structural analyses by finite elements.[15] To explain this technique, consider a set of equations $G(U,P) = 0$ governing the responses $U = \{U_i,\ i = 1, \ldots, n_1\}$ of a structure subjected to a set of external loads $P = \{P_i,\ i = 1, \ldots, n_2\}$. Let $\delta U = \{\delta U_i,\ i = 1, \ldots, n_1\}$ be the additional responses due to a small increment of loads $\delta P = \{\delta P_i,\ i = 1, \ldots, n_2\}$. We can restrict δP such that the additional responses are essentially linear. With this procedure, the nonlinear responses can be calculated by a series of linear analyses. The relations between the previous responses and loads and the additional responses and loads can be expressed as

$$\frac{\partial G}{\partial U}\bigg|_{i-1} \delta U_i + \frac{\partial G}{\partial P}\bigg|_{i-1} \delta P_i = 0 \tag{3}$$

where the $i - 1$ index indicates the previous system while i is for the ith increments of responses and loads.

The nonlinear structural analysis must be initiated with a set of small loads from the zero load configuration. Since explicit mooring line solutions are available for some loading conditions, the incremental process can start from a configuration due to the dominant load. Frequently the other loads are relatively small and the problem can be accurately solved by a single increment. For most mooring line problems the normal hydrodynamic load dominates while the tangential hydrodynamic load and buoyant weight remain relatively small. We shall consider this case. Equations similar to those to be treated here can be readily derived for problems in which the buoyant weight is the dominant force.

Let X, Y, Φ, T represent the first state of the nonlinear system, i.e., a mooring line subjected to normal hydrodynamic load only. The solution for this case with $D(R,\Phi) = R\sin^2\Phi$, including the effect of stretching, can be expressed as

$$\begin{aligned} T &= T_0 \\ S &= \left[T_0/R\left(1 + \frac{T_0}{AE}\right)\right](\cot\Phi_0 - \cot\Phi) \\ X &= (T_0/R)\left[\frac{1}{\sin\Phi_0} - \frac{1}{\sin\Phi}\right] + X_0 \\ Y &= (T_0/R)\left[\ln\tan\frac{\Phi}{2} - \ln\tan\frac{\Phi_0}{2}\right] + Y_0 \end{aligned} \tag{4}$$

These correspond to the solution of Eqs. (1) and (2) with no inertia terms.

Let h_α be a small segment along the cable between points α and $\alpha - 1$. Then the static form of Eqs. (1) can be reduced to

$$\begin{aligned} T_\alpha &= T_{\alpha-1} + \left[W\sin\Phi_{\alpha-1} - F(R_\alpha,\Phi_{\alpha-1})\left(1 + \frac{T_{\alpha-1}}{AE}\right)\right]h_\alpha \\ \Phi_\alpha &= \Phi_{\alpha-1} + \frac{1}{T_{\alpha-1}}\left[D(R_\alpha,\Phi_{\alpha-1})\left(1 + \frac{T_{\alpha-1}}{AE}\right) + W\cos\Phi_{\alpha-1}\right]h_\alpha \end{aligned} \tag{5}$$

Loads taken into account in Eqs. (5), yet neglected in Eqs. (4), are then considered as increments. Consider the case of $D(R_\alpha,\Phi_{\alpha-1}) = R\sin^2\Phi_{\alpha-1}$, $F(R_\alpha,\Phi_{\alpha-1}) = fR$, where f is a scalar factor. Since Eqs. (4) do not consider the buoyant weight and the tangential hydrodynamic load fR, the forces and fR are introduced through the incremental loading procedure. Thus, the incremental loads are taken as $\delta D = 0$, $\delta W = w$, and $\delta F = fR$. Application of Eq. (3) to Eqs. (5) gives

$$\Delta t_\alpha = \Delta t_{\alpha-1} + h_\alpha w\sin\Phi_{\alpha-1} - fR\left(1 + \frac{T_{\alpha-1}}{AE}\right)h_\alpha \tag{6}$$

$$\phi_\alpha = \phi_{\alpha-1} + \frac{h_\alpha}{T_{\alpha-1}} w\cos\Phi_{\alpha-1} - \Delta t_{\alpha-1}\frac{h_\alpha}{T_{\alpha-1}^{\ 2}}R\sin^2\Phi_{\alpha-1} + \frac{h_\alpha}{T_{\alpha-1}}\left(1 + \frac{T_{\alpha-1}}{AE}\right)\phi_{\alpha-1}R\sin 2\Phi_{\alpha-1} \tag{7}$$

where Δt, ϕ have replaced δT, $\delta\Phi$. Any other forms of $F(R,\Phi)$ and $D(R,\Phi)$ are acceptable in this formulation, including concentrated loads and weights. For example, if D in the above case is given by $D(R_\alpha,\Phi_{\alpha-1}) = R(\sin^2\Phi_{\alpha-1} + b\cos\Phi_{\alpha-1})$, where b is a constant, then $\delta D = Rb\cos\Phi_{\alpha-1}$, $\delta W = w$, $\delta F = fR$. The term

$$(\phi_{\alpha-1}/T_{\alpha-1})(1 + T_{\alpha-1}/AE)h_\alpha b\cos\Phi_{\alpha-1}$$

would appear on the right hand side of Eq. (7).

The components of the position vector can now be calculated as

$$\begin{aligned} X_n + x_n &= \sum_{\alpha=1}^{n}\cos(\Phi_\alpha + \phi_\alpha)\left(1 + \frac{T_\alpha + \Delta t_\alpha}{AE}\right)h_\alpha \\ Y_n + y_n &= \sum_{\alpha=1}^{n}\sin(\Phi_\alpha + \phi_\alpha)\left(1 + \frac{T_\alpha + \Delta t_\alpha}{AE}\right)h_\alpha \end{aligned} \tag{8}$$

where n is the number of line segments of length h_α needed to reach the location at which $X + x$ and $Y + y$ are to be calculated.

If w and fR are large in comparison with R, it may be necessary to divide them into several increments and use $T_\alpha + t_\alpha$, $\Phi_\alpha + \phi_\alpha$ obtained from Eqs. (6) and (7) as the prestressed configuration. This process can be repeated as many times as necessary. For most mooring line static problems one increment suffices.

For convenience, Eqs. (6) and (7) can be written in the line solution or transfer matrix form

$$\begin{bmatrix}\Delta t \\ \phi \\ 1\end{bmatrix}_\alpha = \begin{bmatrix}1 & 0 & A_3 \\ A_1 & A_2 & A_4 \\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\Delta t \\ \phi \\ 1\end{bmatrix}_{\alpha-1} \tag{9}$$

where

$$A_1 = -\frac{R}{T_{\alpha-1}^{\ 2}}h_\alpha\sin^2\Phi_{\alpha-1},$$

$$A_2 = 1 + \frac{R}{T_{\alpha-1}}\left(1 + \frac{T_{\alpha-1}}{AE}\right)h_\alpha\sin 2\Phi_{\alpha-1},$$

$$A_3 = h_\alpha w\sin\Phi_{\alpha-1} - fR\left(1 + \frac{T_{\alpha-1}}{AE}\right)h_\alpha,$$

$$A_4 = \frac{h_\alpha}{T_{\alpha-1}}w\cos\Phi_{\alpha-1}$$

or

$$\bar{V}_\alpha = L_{\alpha-1}\bar{V}_{\alpha-1} \tag{10}$$

where $L_{\alpha-1}$ is the 3×3 transfer matrix in Eq. (9) and

$$\bar{V}_\alpha = \{\Delta t_\alpha, \phi_\alpha, 1\}, \quad \bar{V}_{\alpha-1} = \{\Delta t_{\alpha-1}, \phi_{\alpha-1}, 1\}.$$

The progressive multiplication characterizing the line solution approach gives

$$\bar{V}_\alpha = L_{\alpha-1}L_{\alpha-2} \cdots L_0\bar{V}_0 \tag{11}$$

and for $\alpha = n$ at the end

$$\bar{V}_n = L_{n-1}L_{n-2} \cdots L_0\bar{V}_0 \tag{12}$$

This final equation relates the parameters or state variables ϕ_α, Δt_α at any point to the parameters at one end of the line. For given boundary conditions, Δt_n, ϕ_n, Δt_0, and ϕ_0 can be obtained without iteration.

Varying Properties along the Length

Mooring lines may have variable cross sections or may even be made of different materials along the length. These can be taken into consideration simply by inserting different stiffness values in the appropriate transfer matrices.

Concentrated Loads

Concentrated loads of small magnitude can be taken into account through a transfer matrix similar to Eq. (9). If H_α and V_α are the horizontal and vertical components of the concentrated loads at α, then the transfer or point matrix is

$$\begin{bmatrix} \Delta t \\ \phi \\ 1 \end{bmatrix}_{\alpha+0} = \begin{bmatrix} 1 & 0 & -(V_\alpha \sin\Phi_\alpha - H_\alpha \cos\Phi_\alpha) \\ 0 & 1 & (1/T_\alpha)(H_\alpha \sin\Phi_\alpha - V_\alpha \cos\Phi_\alpha) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta t \\ \phi \\ 1 \end{bmatrix}_{\alpha-0} \tag{13}$$

If the loads are large, they should be considered in the nonlinear analysis. Here, apply Eq. (10) from 0 to $\alpha - 0$ for $T_{\alpha-0}$, $\Phi_{\alpha-0}$. Then

$$\begin{aligned} \Phi_{\alpha+0} &= \Phi_{\alpha-1} + (1/T_\alpha)(H_\alpha \sin\Phi_\alpha - V_\alpha \cos\Phi_\alpha) \\ T_{\alpha+0} &= T_{\alpha-1} - (V_\alpha \sin\Phi_\alpha - H_\alpha \cos\Phi_\alpha) \end{aligned} \tag{14}$$

where V_α and H_α are positive in the positive x and y directions, respectively.

Dynamic Analysis

The incremental linearization method can also be used for a dynamic analysis. Let T, Φ, X, Y represent the static equilibrium condition and $\Delta t(t)$, $\phi(t)$, $x(t)$, $y(t)$ be the motion due to the change of current $v(t)$ or any other dynamic load. From the compatibility and equilibrium conditions of the mooring lines, if only first-order terms are retained, we find

$$\text{Normal drag} = 1/2\, C_n[V^2 + 2Vv(t)] = R + C_n Vv(t)$$

$$\begin{aligned} D &= [R + C_nVv(t)] \sin^2(\Phi + \phi(t)) \\ &= R \sin^2\Phi + C_nVv(t) \sin^2\Phi + \phi(t)R \sin 2\Phi \end{aligned} \tag{15}$$

$$F = fR + fC_nVv(t) \tag{16}$$

It follows from Eqs. (8) that

$$\begin{aligned} x_n(t) &= -\sum_{\alpha=1}^{n} \phi_{\alpha-1}(t)\left(1 + \frac{T_{\alpha-1}}{AE}\right)h_\alpha \sin\Phi_{\alpha-1} \\ y_n(t) &= \sum_{\alpha=1}^{n} \phi_{\alpha-1}(t)\left(1 + \frac{T_{\alpha-1}}{AE}\right)h_\alpha \cos\Phi_{\alpha-1} \end{aligned} \tag{17}$$

The inertia forces at α due to the small motion are then

$$\begin{aligned} \left(\frac{w}{g} + m_{xx}\right)\ddot{x} &= \\ -\left(\frac{W}{g} + m_{xx}\right)&h_\alpha \sum_{i=1}^{\alpha} \ddot{\phi}_{i-1}(t)\left(1 + \frac{T_{i-1}}{AE}\right)h_i \sin\Phi_{i-1} \\ \left(\frac{w}{g} + m_{yy}\right)\ddot{y} &= \\ \left(\frac{W}{g} + m_{yy}\right)&h_\alpha \sum_{i=1}^{\alpha} \ddot{\phi}_{i-1}(t)\left(1 + \frac{T_{i-1}}{AE}\right)h_i \cos\Phi_{i-1} \end{aligned} \tag{18}$$

If terms including $v(t)$, $\ddot{\phi}(t)$ are treated as load increments, we obtain equations similar to Eqs. (6) and (7).

$$\begin{aligned} \Delta t_\alpha = {} & \Delta t_{\alpha-1}(t)\left(1 - Rf\frac{h_\alpha}{AE}\right) \\ & + \phi_{\alpha-1}(t)wh_\alpha \cos\Phi_{\alpha-1} - fC_nVv(t)h_\alpha\left(1 + \frac{T_{\alpha-1}}{AE}\right) \\ & + h_\alpha\left(\frac{w}{g} + m_{xx}\right)\sum_{i=1}^{\alpha} \ddot{\phi}_{i-1}(t)\left(1 + \frac{T_{i-1}}{AE}\right)h_i \\ & \qquad \sin(\Phi_{i-1} - \Phi_{\alpha-1}) \end{aligned} \tag{19}$$

$$\begin{aligned} \phi_\alpha(t) = {} & [1 - h_\alpha w \sin\Phi_{\alpha-1} - \frac{h_\alpha}{T_{\alpha-1}^2} R \sin 2\Phi_{\alpha-1} \\ & + \frac{2h_\alpha}{T_{\alpha-1}} R \cos 2\Phi_{\alpha-1}\left(1 + \frac{T_{\alpha-1}}{AE}\right)]\phi_{\alpha-1}(t) \\ & - \frac{h_\alpha}{T_{\alpha-1}^2} R \sin 2\Phi_{\alpha-1}\Delta t(t) \\ & - h_\alpha\left(\frac{w}{g} + m_{yy}\right)\sum_{i=1}^{\alpha} \ddot{\phi}_{i-1}\left(1 + \frac{T_{i-1}}{AE}\right)h_i \\ & \qquad \cos(\Phi_{\alpha-1} - \Phi_{i-1}) \\ & + h_\alpha C_nVv(t) \sin^2\Phi_{\alpha-1} \end{aligned} \tag{20}$$

For harmonic motion let

$$\phi_\alpha = \phi_\alpha^* \sin\omega t$$
$$\Delta t_\alpha = \Delta t_\alpha^* \sin\omega t \quad \text{for all } \alpha \tag{21}$$
$$v = v^* \sin\omega t$$

Substitution of Eqs. (21) into Eqs. (19) and (20) yield time independent relations in ϕ_α^*, Δt_α^*, v^* similar to Eqs. (6) and (7). By the same steps as those from Eqs. (6) to (12) we have

$$\begin{bmatrix} \Delta t^* \\ \phi^* \\ 1 \end{bmatrix}_n = \begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta t^* \\ \phi^* \\ 1 \end{bmatrix}_0 \tag{22}$$

where L_{ij} are the elements of the overall transfer matrix. In this particular case L_{13}, L_{23} involve terms with v^* only.

To find the natural frequencies, set $v_\alpha = 0$ for all α. Equations (22) reduce to

$$\begin{bmatrix} \Delta t^* \\ \phi^* \end{bmatrix}_n = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} \Delta t^* \\ \phi^* \end{bmatrix}_0 \tag{23}$$

The natural frequencies can be computed from Eq. (23) for given boundary conditions. If the tension and the inclination at n are the same as for the static equilibrium condition, then $\Delta t_n^* = \phi_n^* = 0$ and the frequency equation is given by

$$\det \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} = 0 \tag{24}$$

If it is required that the tensions at both ends are to be kept the same as in the static equilibrium condition, then

$$L_{12} = 0 \tag{25}$$

is the frequency equation.

Outline of the Static Analysis

1) Consider first a problem in which the tension T and angle of inclination Φ are prescribed at one point. Assume the length of mooring line is sought for a specified depth of water. a) Treat the mooring line as though it is acted upon by only normal hydrodynamic loads. b) The mooring line is then divided into n segments whose lengths are not necessarily uniform. The values of Φ_α, T_α are calculated for each nodal point using Eq. (4) with $S = S_\alpha$. c) Form the transfer matrices for each segment as indicated in Eq. (9). The boundary conditions to be applied to Δt, ϕ are found from known physical conditions on the ends. For example, suppose $\bar{T}$ and $\bar{\Phi}$ are given at one end in the form $\Phi_0 + \phi_0 = \bar{\Phi}_0$, $T_0 + \Delta t_0 = \bar{T}_0$. We have from step b, $\Phi_0 = \bar{\Phi}_0$, $T_0 = \bar{T}_0$; hence, $\phi_0 = \Delta t_0 = 0$. With these conditions, Δt_α. ϕ_α can be obtained for all α, $\alpha = 1, 2, \ldots, n$. d) Substitute T_α, Δt_α, ϕ_α, h_α into Eq. (8) and take the summation until $Y + y$ is equal to the specified depth. In general, it is necessary to take a fraction, r, of the length of some segment h_k such that the required depth is equal to

$$\sum_{\alpha=1}^{k} \sin(\Phi_\alpha + \phi_\alpha)\left(1 + \frac{T_\alpha + \Delta t_\alpha}{AE}\right)\bar{h}_\alpha \qquad k \le n$$

$$\bar{h}_\alpha = \begin{cases} h_\alpha & \text{for } \alpha < k \\ h_\alpha r & \text{for } \alpha = k \end{cases}$$

The required length of the mooring line is

$$L = \sum_{\alpha=1}^{k} \bar{h}_\alpha$$

2) As a second problem, assume the length L of the mooring line is given along with the locations of the end points. The tension and angle of inclination are sought. a) Calculate T_0, Φ_0 from Eqs. (4). b) Divide the mooring line into n segments and calculate T_α, Φ_α, for all α. c) Form all transfer matrices [Eq. (9)]. d) Express Δt_α, ϕ_α in terms of Δt_0, ϕ_0 using Eq. (11) and substitute Δt_α, ϕ_α, Φ_α, T_α into Eqs. (8). This generates two equations for Δt_0 and ϕ_0. Solve for Δt_0, ϕ_0, and compute Δt_α, ϕ_α using Eq. (11) or Eqs. (6) and (7). Then the total tension is $T_\alpha + \Delta t_\alpha$ and the angle is $\Phi_\alpha + \phi_\alpha$.

Example of a Static Analysis

Consider a problem treated by Pode.[2] It is desired to anchor a buoy in 3600 ft of water using a 7/16 in. diam stranded cable. The cable weighs 0.27 lb/ft in water. The drag of the cable when normal to the stream at five knots is 3.9 lb/ft. The buoy has an excess buoyancy of 7300 lb when fully submerged and in this condition in a current of five knots it has a dynamic lift of 1800 lb and a drag of 5200 lb. What is the minimum length of cable required to insure that the buoy will never be submerged if the ocean currents are always uniform and less than five knots?

At point 2 of Fig. 3, we know the tension is given by $T_2 = (9100^2 + 5200^2)^{1/2} = 10{,}500$ lb and the angle of inclination is $\tan\phi_2 = 9100/5200 = 1.75$ or $\phi_2 = 60.258°$.

Stiff Line

If the stretch of the line is neglected, we find from Eq. (12) $\phi_1 = 17.36°$, $T_1 = 10{,}500$ lb. The boundary conditions for ϕ, Δt are $\phi_2 = 0$, $\Delta t_2 = 0$. Since we begin at the top, we use Eqs. (8) with opposite signs assigned to the terms on the left hand side of the first two equations. We stop at

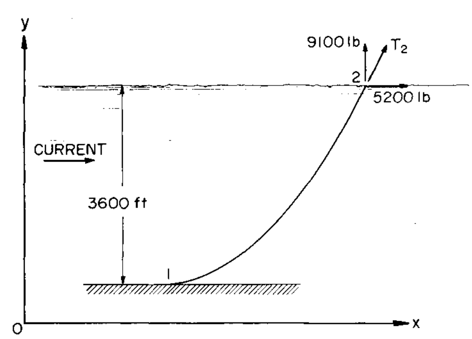

Fig. 3 Buoy anchor.

Table 1 Results of example problem

Parameter	Pode's method	Present method: No stretch	Present method: Nylon AE = 1.43×10^5
ϕ_2	60.25°	60.25°	60.25°
T_2	10,500 lb	10,500 lb	10,500 lb
ϕ_1	8.63°	8.67°	9.48°
T_1	10,240 lb	10,222 lb	10,250 lb
S	8,700′	8,750′	8,050′
$x_2 - x_1$	7,640′	7,620′	7,454′

$y = 3600$ ft. The results are

$$\phi_1 = -8.67°, \; \Phi_1 + \phi_1 = 8.67°;$$

$$\Delta t_1 = -278 \text{ lb}, \; T_1 + \Delta t_1 = 10{,}222 \text{ lb};$$

$$S = 8750 \text{ ft}; \quad x_2 - x_1 = 7620 \text{ ft}$$

Nylon Cable

In the case of a nylon line with AE = 143,000 lb, we find

$$\phi_1 = -7.88°, \; \Phi_1 + \phi_1 = 9.48°;$$

$$\Delta t_1 = -250 \text{ lb}, \; T_1 + \Delta t_1 = 10{,}250 \text{ lb};$$

$$S = 8050 \text{ ft}; \quad x_2 - x_1 = 7450 \text{ ft}$$

The results are summarized and compared with Pode's solution in Table I.

Outline of the Dynamic Analysis

a) Compute the static equilibrium configuration of the mooring line.

b) For harmonic loads, treat ϕ_α^*, Δt^* as increments of responses and the terms with v^* as increments of loads.

c) For natural frequencies, substitute Δt^*, ϕ^* for $\Delta t(t)$, $\phi(t)$ and $-\omega^2\phi^*$ for $\ddot{\phi}(t)$ into Eqs. (19) and (20) and set $v = 0$. Then use a search procedure to find that ω^2 satisfies the prescribed boundary conditions.

Summary

An efficient technique for the static and dynamic analysis of mooring lines has been presented. This approach avoids the limiting restrictions and overcomes most of the difficulties inherent to other approaches. The coupling of an incremental linearization with the line solution approach permits a mooring line with arbitrary geometric configuration and physical properties to be treated.

References

[1]Beateaux, H. O., "Design of Deep Sea Mooring Lines," *Marine Technology Society Journal,* Vol. 4, No. 3, May/June 1970, pp. 33–46.

[2]Casarella, M. J. and Parsons, M., "Cable System Under Hydrodynamic Loadings," *Marine Technology Society Journal,* Vol. 4, No. 4, July/Aug. 1970, pp. 27–44.

[3]Pode, L., "Tables for Computing the Equilibrium Configuration of a Flexible Cable in a Uniform Stream," Rept. 687, March 1951, David Taylor Model Basin, Washington, D. C., Supplement published Sept. 1955.

[4]Li, W., "Elastic Flexible Cable in Plane Motion under Tension," *Journal of Applied Mechanics,* Vol. 81, No. 3, Dec. 1959, pp. 587–593.

[5]Davenport, A. G. and Steels, G. N., "Dynamic Behavior of Massive Guy Cables," *Transactions of the ASCE: Journal of the Structural Division,* Vol. 81, No. 2, April 1955, pp. 43–70.

[6]Soler, A. I., "Dynamic Response of Single Cables with Initial Sag," *Journal of the Franklin Institute,* Vol. 290, No. 4, Oct. 1970, pp. 377–387.

[7]Alexandrov, M., "On the Dynamics of Cables with Application to Marine Use," *Marine Technology,* Vol. 8, No. 1, Jan./Feb./March 1971, pp. 84–92.

[8]Thornton, C. H. and Birnstiel, C., "Three-Dimensional Suspension Structures," *Transactions of the ASCE: Journal of the Structural Division,* Vol. 93, No. 2, April 1967, pp. 247–270.

[9]Baron, F. and Venkatesan, M. S., "Nonlinear Analysis of Cable and Truss Structures," *Transactions of the ASCE: Journal of the Structure Division,* Vol. 97, No. 2, Feb. 1971, pp. 679–710.

[10]Skop, R. A. and O'Hara, G. J., "The Static Equilibrium Configuration of Cable Arrays by Use of the Method of Imaginary Reactions," NRL-6819, Feb. 1969, Naval Research Lab., Washington, D. C.

[11]Walton, T. S. and Polachek, H., "Calculation of Transient Motion of Submerged Cables," *Mathematical Tables and Aids to Computation, American Mathematical Society,* Vol. 14, 1960, pp. 27–60.

[12]Nath, J. H. and Felix, M. P., "Dynamics of a Single Point Mooring in Deep Water," *Proceedings of the ASCE, Civil Engineers in the Ocean,* Vol. II, 1969, pp. 45–64.

[13]Reid, R. O., "Dynamics of Deep-Sea Mooring Lines," Rept., July 1968, Dept. of Oceanography, Texas A & M Univ., College Station, Texas.

[14]Schneider, L. and Nickels, F., "Cable Equilibrium Trajectory in a Three-Dimensional Flow Field," *Transactions of the ASME: Winter Annual Meeting,* Paper 66-WA/UNT-12, Nov. 1966.

[15]Oden, J. T., "Finite Element Applications in Nonlinear Structural Analysis," *Proceedings of the ASCE: Symposium on Applications of Finite Element Methods in Civil Engineering,* 1969, pp. 419–456.